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REPRESENTING HOMOLOGY CLASSES OF $S^2 \times S^2$

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§1. INTRODUCTION

LET S^2 denote the 2-sphere. The purpose of this note is to prove the following.

THEOREM 1. *Let ξ_1 and ξ_2 be natural generators of $H_2(S^2 \times S^2; \mathbb{Z})$. Then $\gamma = p\xi_1 + q\xi_2$, $p, q \in \mathbb{Z}$, can be represented by a smoothly embedded 2-sphere in $S^2 \times S^2$ if and only if $|p| \leqq 1$ or $|q| \leqq 1$.*

Addendum. In the situation above, any smoothly immersed 2-sphere representing γ in $S^2 \times S^2$ has a self-intersection whose sign $= \operatorname{sign}(pq) \cdot \operatorname{sign}(\xi_1 \cdot \xi_2)$ when $|p| \geqq 2$ and $|q| \geqq 2$.

Remark 1. Note that the "if" part of Theorem 1 is trivial. Also note that there is a smooth immersion of S^2 representing γ which has no self-intersection point of $(-1) \cdot \operatorname{sign}(pq) \cdot \operatorname{sign}(\xi_1 \cdot \xi_2)$, ($|p| \geqq 2$, $|q| \geqq 2$). Figure 1 below illustrates how to immerse $3\xi_1 + 2\xi_2$ without having negative (positive) self-intersection points.

Remark 2. If p and q have a common divisor $\neq 1$ then Theorem 1 is known. (See Tristram[7], Hsiang–Szczarba[3], Rohlin[6].)

Remark 3. Note that it is not hard to see that any class in $H_2(S^2 \times S^2; \mathbb{Z})$ can be realized by a PL embedded 2-sphere possibly with one non-locally-flat point. (See Kervaire–Milnor[4].)

Remark 4. If p and q are relatively prime integers. Then $\gamma = p\xi_1 + q\xi_2$ is realized by (4-ball)U(one Casson handle), which is a neighborhood of a TOP embedded locally flat 2-sphere by Freedman[2]. Hence smoothness condition in Theorem 1 is essential.

After changing the orientation of $S^2 \times S^2$ and/or the orientation of ξ_i if necessary, Theorem 1 is a special case of the following.

THEOREM 2. *Let M^4 be a closed* 1-*connected* DIFF 4-*manifold with the intersection form*

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus \langle 1 \rangle \oplus \langle 1 \rangle \oplus \cdots \oplus \langle 1 \rangle,$$

w.r.t. *generators $\xi_1, \xi_2, \zeta_1, \zeta_2, \ldots \zeta_m$ of $H_2(M^4; \mathbb{Z}) \cong \oplus^{m+2}\mathbb{Z}$, $(m \geqq 0)$. Then the homology class $\gamma = p\xi_1 + q\xi_2 + \Sigma_{i=1}^n \zeta_i$, $(0 \leqq n \leqq m)$ cannot be represented by a smoothly embedded 2-sphere in M^4 provided $p \geqq 2$, $q \geqq 2$, and $0 \leqq n \leqq 2pq - 1$.*

Remark 5. The DIFF 4-manifold M^4 above is actually DIFF h-cobordant to $S^2 \times S^2 \# (m\mathbb{C}P^2)$, hence, by Freedman[2], TOP homeomorphic to it. However, the DIFF structure might be non-standard.

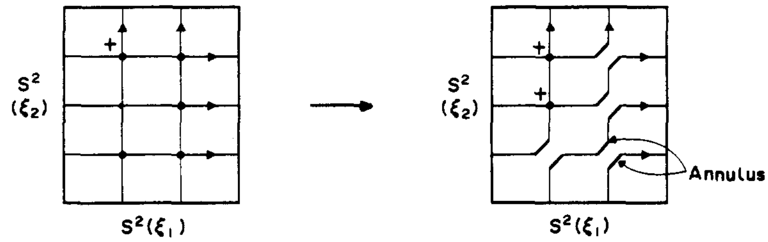

Fig. 1.

We shall use the following result of Donaldson[1]: if the intersection form of a closed 1-connected smooth 4-manifold is positive definite, then this form is standard, i.e. equivalent over $\mathbb{Z}$ to $\langle 1\rangle \oplus \langle 1\rangle \oplus \ldots \oplus \langle 1\rangle$.

Professor A. J. Casson mentioned to me that this result can detect the DIFF non-sliceness of some TOP slice knots (e.g. the pretzel knot $K(-3, 5, 7)$). The proof of our theorem is essentially based on his idea in this non-sliceness problem. We would also like to thank him for many valuable discussions.

§2. PROOF OF THEOREM 2

Suppose $\gamma = p\xi_1 + q\xi_2 + \Sigma_{i=1}^{n} \zeta_i$, $p \geq 2$, $q \geq 2$, is represented by a smoothly embedded 2-sphere S in M^4. Adding $\#(\mathbb{C}P^{2}$'s) to M^4 and $\#(\mathbb{C}P^{1}$'s) $\subset \#(\mathbb{C}P^{2}$'s) to S if necessary, we may assume $n = 2pq - 1 \leqq m$. Then the self-intersection of $S = S \cdot S = -2pq + (2pq - 1) = -1$. Since D^2-bundles over S^2 are classified by the Euler number $\in \mathbb{Z} \cong \pi_1(SO(2))$, the tubular neighborhood N of S in M^4 is the (-1)-Hopf bundle over S. Hence we can blow down S and obtain a closed smooth 4-manifold W^4, i.e. noting that the boundary of N is S^3, set $W^4 = (M^4 - \operatorname{Int} N) \cup_\partial D^4$.

Claim 1. W^4 is a closed 1-connected smooth 4-manifold with a positive definite intersection form.

Proof. W^4 is simply-connected, since $\pi_1(M^4 - \operatorname{Int} N)$ is generated by a meridional circle to S, which lies on the boundary of the 4-ball D^4 in W^4. Note that $M^4 = W^4 \# \hat{N}$, where $\hat{N} \cup_\partial D^4 \cong \overline{\mathbb{C}P^2}$. Hence the intersection form of M^4 is the direct sum of the intersection form of W^4 and that of $\hat{N} \cong \overline{\mathbb{C}P^2}$. Diagonalizing the intersection matrix of M^4 using coefficient $\mathbb{R}$, we may see the intersection form restricted to $H_2(W^4)$ is positive definite. $\square$

Let α and β be two integer vectors in $H_2(W^4; \mathbb{Z})$ such that $\alpha \cdot \alpha = \beta \cdot \beta = 1$. Then, by the positive definiteness, $(\alpha - \beta) \cdot (\alpha - \beta) = 2(1 - \alpha \cdot \beta) \geqq 0$, $(\alpha + \beta) \cdot (\alpha + \beta) = 2(1 + \alpha \cdot \beta) \geqq 0$, where $1 - \alpha \cdot \beta = 0$ iff $\alpha - \beta = 0$, and $1 + \alpha \cdot \beta = 0$ iff $\alpha + \beta = 0$. This implies $\alpha \perp \beta$ or $\alpha = \pm \beta$. Hence all α in $H_2(W^4; \mathbb{Z})$ with $\alpha \cdot \alpha = 1$ form (up to sign) the orthonormal basis for the standard part of $H_2(W^4; \mathbb{Z})$. Especially the intersection form of W_4 is standard, i.e. equivalent to $\langle 1\rangle \oplus \langle 1\rangle \oplus \ldots \oplus \langle 1\rangle$, iff the number of such α's $= 2 \cdot \operatorname{rank} H_2(W^4; \mathbb{Z}) = 2(m + 1)$.

If we write $\alpha = x\xi_1 + y\xi_2 + \Sigma_{i=1}^{2pq-1} z_i\zeta_i + \Sigma_{j=2pq}^{m} w_j\zeta_j$, then such α's are obtained by solving the simultaneous Diophantine equation

$$\left.\begin{aligned} qx + py &= \sum_{i=1}^{2pq-1} z_i \\ 1 + 2xy &= \sum_{i=1}^{2pq-1} z_i^2 + \sum_{j=2pq}^{m} w_j^2, \end{aligned}\right\} \qquad (1)$$

where the first equation is equivalent to $\alpha \in H_2(W^4; \mathbb{Z})$, which is the orthogonal space to γ, and the second equation is equivalent to $\alpha \cdot \alpha = 1$.

Claim 2. The Diophantine equation (1) has exactly $2(m - 2pq + 1)$ solutions provided $p \geqq 2$ and $q \geqq 2$.

Proof. First note that (1) has $2(m - 2pq + 1)$ trivial solutions, namely $x = y = z_i = 0$, $w_j = \pm \delta_{jk}$ (Kronecker's delta), $2pq \leqq k \leqq m$. Hence w_j-coordinates of the remaining solutions are zero. Hence it suffices to show that the equation

$$\left.\begin{aligned} qx + py &= \sum_{i=1}^{2pq-1} z_i \\ 1 + 2xy &= \sum_{i=1}^{2pq-1} z_i^2 \end{aligned}\right\} \tag{2}$$

has no integer solution when $p \geqq 2$, $q \geqq 2$.

Discarding z_i's which are zero and renumbering, we may assume $qx + py = \Sigma_{i=1}^r z_i$, $1 + 2xy = \Sigma_{i=1}^r z_i^2$, and each $z_i \neq 0$ $(1 \leqq i \leqq r)$, for some $0 \leqq r \leqq 2pq - 1$. Then $2pq \cdot 2xy \leqq (qx + py)^2 = (\Sigma_{i=1}^r z_i)^2 \leqq r(\Sigma_{i=1}^r z_i^2) = r(1 + 2xy)$. Hence $2xy \leqq r/(2pq - r) \leqq r$. Hence we have

$$\sum_{i=1}^{r} z_i^2 \leqq r + 1. \tag{3}$$

This implies $z_i = \pm 1$, $(1 \leqq i \leqq r)$. Suppose there are $(r - s)$1's and $s(-1)$'s among z_i's $(0 \leqq s \leqq r)$. The equation now becomes

$$\left.\begin{aligned} qx + py &= r - 2s \\ 1 + 2xy &= r. \end{aligned}\right\} \tag{4}$$

Eliminating y we have a quadratic equation w.r.t. x: $2qx^2 - 2(r - 2s)x + p(r - 1) = 0$, whose $d = (\text{discriminant})/4 = (r - 2s)^2 - 2pq(r - 1) \leqq (r - 2s)^2 - (r + 1)(r - 1) = 1 - 4s(r - s)$, (Note that $r \neq 0$). Hence

$$0 \leqq d \leqq 1 - 4s(r - s), \tag{5}$$

which implies $s = 0$ or r, and $d = 0$ or 1.

We may w.l.o.g. assume $s = 0$. If $d = 0$, then $r^2 - 2pqr + 2pq = 0$ and $r = pq \pm ((pq - 1)^2 - 1)^{1/2}$, which cannot be an integer $(pq \geqq 4)$.

If $d = 1$, then we have $r = 1$ or $2pq - 1$.

if $r = 1$, then the equation (4) becomes

$$\left.\begin{aligned} qx + py &= 1 \\ xy &= 0, \end{aligned}\right\} \tag{6}$$

which has no solution when $p \geqq 2$, $q \geqq 2$.

If $r = 2pq - 1$, then setting $x' = p - x$, and $y' = q - y$, we have the same equation as (6). $\square$

Claim 2 implies that the intersection form of our W^4 is non-standard. This contradicts

Donaldson's result[1], which says the intersection form of W^4 as in Claim 1 is always standard. This completes the proof of Theorem 2.

§3. PROOF OF ADDENDUM

Changing the orientation of $S^2 \times S^2$ and/or the orientation of ξ_i appropriately we may assume $p \geq 2$, $q \geq 2$, and $\xi_1 \cdot \xi_2 = -1$. Then $\operatorname{sign}(pq)\operatorname{sign}(\xi_1 \cdot \xi_2) = -1$. Hence the following claim implies Addendum.

Claim 3. In the situation of Theorem 2, any smoothly immersed 2-sphere S representing γ in M^4 has a negative self-intersection.

Proof. Let P be a transverse self-intersection point of S. Suppose $\operatorname{sign}(P) = +1$. Let C and $\bar{C}$ be two copies of $\mathbb{C}P^1$ in $\mathbb{C}P^2$, meeting transversely in one point Q, and oriented oppositely. Then by a Norman trick (see [5]) in $M^4 \# \mathbb{C}P^2$ using C and $\bar{C}$, along an arc A from P to Q, i.e. by "double" connected sum $S \# C \# \bar{C}$ in $M^4 \# \mathbb{C}P^2$ along A, as illustrated in Fig. 2, we can remove the positive intersection point P from S without changing the homology class $\gamma = [S] = [S \# C \# \bar{C}]$. (Note that we used $\operatorname{sign}(Q) = -1$.)

Hence we can remove all positive self-intersection points from S in $M^4 \# k\mathbb{C}P^2$ for sufficiently large k. Note that Theorem 2 implies that we cannot eliminate all the self-intersection points of S even in the stabilized manifold $M^4 \# (\mathbb{C}P^2\text{'s})$. Hence S has a negative self-intersection point. □

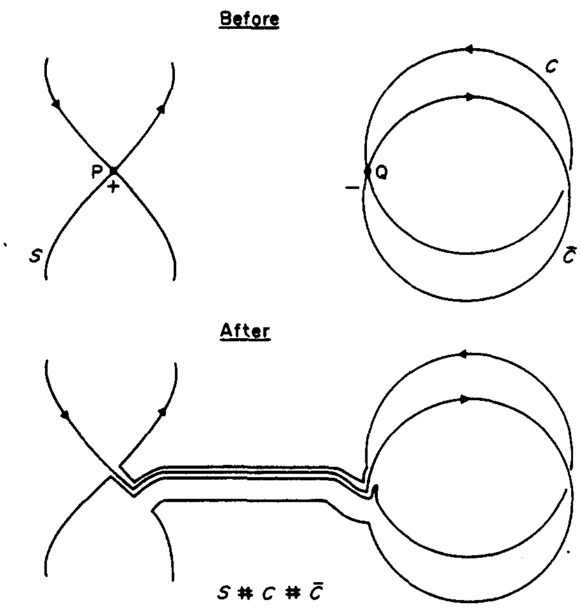

Fig. 2. Norman trick in $M^4 \# \mathbb{C}P^2$.

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